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Methods of Fractal Dimension Computation

Prof. A. Annadhason,
Head & Asst. Professor of Computer Science Department, St. Jude's College, India.

Abstract: The different methods for calculating fractal dimension are walking-divider method, Box counting method, Prism counting method, Epsilon-Blanket method, Perimeter-Area Relationship, Fractional Brownian Motion, Power spectrum method and Hybrid method .

***Key words: Step, Length,* dimension**

1.1 Walking-Divider method

This method uses a chord length *(Step)* and measures the number of chord lengths *(Length)* required to cover a fractal curve. The technique is based on the principle of taking rulers of varying size (*Step)* to cover the curve and counting the number of rulers *(Length)* required in each case. $N(\delta) = Length$ and $\delta = Step$ are estimated. A least squares fit to the bilogarithmic plot of *Length* against *Step* gives the slope β which is the fractal dimension, D with sign reversed.

$D = -\log N(\delta) / \log\delta$ **……………(1.1.1)**

The limitation of this method is that the initial and final *Step* must be carefully chosen. An appropriate starting value suggested by Shelberg (1982) is half of the average distance between the points. Also, the computation of the initial value, and the procedure required to count the number of *Steps,* makes this algorithm time consuming.

1.2 Box Counting

The most popular algorithm for computing the fractal dimension of one dimensional and two dimensional data is the box counting, method originally developed by Voss (1985). In this method, the fractal surface is covered with a grid of n- dimensional boxes or hyper-cubes with side length, δ and counting the number of boxes that contains a part of the fractal $N(\delta)$. For signals, the grid consists of squares and for images, the grid consists of cubes. The fractal surface is covered with boxes of recursively different sizes. An input signal with N elements or an image of size $N * N$ is used as input where N is a power of 2. The slope β is obtained from the logarithmic plot of the number of boxes used to cover the fractal against the box size and the fractal dimension D_B is given by

$D_B = -\beta$ **……………. (1.2.1)**

This dimension is also known as the box or Minkowski dimension. For a smooth one-dimensional curve, it is expected that

$N(\delta) = L/\delta$ ……….. **(1.2.2)**

where L is the length of the curve, $N(\delta)$ is the number of nonempty boxes and δ is the step size. The generalization to the box counting measure is given as

$N(\delta) \quad \alpha \; 1/\delta^{D_B}$ …………… **(1.2.3)**

$D_B = \lim -(\log N(\delta) / \log \delta)$ ..………… **(1.2.4)**

A regular grid is usually applied to the data and the non-empty boxes are counted. The greater the number of points used for the least squares fit, the better the estimate of the fractal dimension. In a two-dimensional version of this algorithm, Sarkar and Chaudhuri (1992) have considered the problem of optimizing the number of boxes for a given size required to compute a fractal dimension close to the Hausdorff dimension.

1.3 Prism Counting

The algorithm which is similar to box counting technique, computes the area based on four triangles defined by the corner points followed by summation over a grey level surface. The triangles define a prism based on the elevated corners and a central point computed in terms of the average of the four corners. A bilogarithmic plot of the sum of the prisms' areas for a given base area gives a fit to a line whose slope is β in which

$D = 2 - \beta$ **……………..(1.3.1)**

Here D is the fractal dimension. Though the algorithm is similar to the box counting method, it is slower due to the number of multiplications implied by the calculation of the areas.

1.4 Epsilon-Blanket

In this method the fractal dimension of curves/surfaces are computed using the area/volume measured at different scales (Peleg et al., 1984). In the case of curves, the set of points whose distance from a curve which is not more than a small scale, ε is considered. This gives a strip of width *2*ε that surrounds the curve. The length of the curve $L(\varepsilon)$ is calculated from the strip area $A(\varepsilon)$ by

$L(\varepsilon) = A(\varepsilon)/2\varepsilon$ **…………...(1.4.1)**

The fractal dimension, *D* is computed using the relation

$L(\varepsilon) \; \alpha \; \varepsilon^{(1-D)}$ **…………(1.4.2)**

In the case of surfaces, the set of points in three-dimensional space which is not more than ε from the surface, gives a 'blanket' of volume $V(\varepsilon)$ whose width is again *2*ε. The surface area is given by

$A(\varepsilon) = V(\varepsilon) / 2\varepsilon$ **…………..(1.4.3)**

and *D* can be computed as
$A(\varepsilon) = \varepsilon^{(2-D)}$ **……………(1.4.4)**

2.1 Perimeter-Area Relationship

Here, the perimeter L is related to the enclosed area *A* for a non-fractal closed curve in the plane by $L = c\sqrt{A}$. Here the pre-factor *c* is a constant for a given shape. For squares *c = 1* and for circles $c - 2\sqrt{\pi}$. The above relation is generalized by Mandelbrot (1983) for closed fractal curves as follows from which the fractal dimension, *D* can be computed as

$L = c (\sqrt{A})^D$ **………….. (2.1.1)**

for $1 < D < 2$

2.2 Fractional Brownian Motion

A fractional Brownian motion *(fB$_m$), $B_H(t)$,* is a function whose increments

$\Delta B_H(X) = B_H(t + x)-B_H(t)$ …………. **(2.2.1)**

has a zero-mean Gaussian distribution with variance given by

$\langle |B_H(t + x) - B_H(t)|^2 \rangle \ \alpha \ |x|^{(2H)}$ **………….. (2.2.2)**

(Turner et al., 1998). The parameter $0 < H < 1$, defines the scaling behaviour. If a fBm that covers a time period $\Delta t = 1$ is considered and the vertical range ΔB_H is defined as one, then $B_H(t)$ is statistically self-similar. So if the time span is divided into $N = l/(\Delta t)$ equal intervals, the vertical range within these intervals will be

$\Delta B_H = \Delta t^H = 1/N^H = N^{(-H)}$ **…………...(2.2.3)**

Using the box counting method, with boxes of length $\delta = N^{-1}$, the number of boxes required to cover each interval is

$\Delta B_H \Delta t = N^{-H}/N^{-1} = N^{(1-H)}$ **…………...(2.2.4)**

which means

$N(\delta) = NN^{(1-H)} = N^{(2-H)}$ **…………(2.2.5)**

So from equation (2.11)
$D = 2 - H$ **……..…..(2.2.6)**

The same formula can be extended to signals of higher topological dimensions to compute the respective fractal dimension.

2.3 Power Spectrum

This method is an application of the Fourier power spectrum method. The real-space image is Fourier transformed by means of the fast fourier transform and the power spectrum, P_i is computed as

$P_i = Re(k_i)^2 + I_m(k_i)^2$ **………….. (2.3.1)**

Then the power spectrum of an ideal one dimensional fractal signal with dimension *D* is considered. This has the formula,

$Pi = c \ kj^{-\beta}$ **…………...(2.3.2)**

where *c* is a constant and β is the spectral exponent. The index β is related to the Fourier transform dimension, D_F.

The value of the spectral β and D_F, can be found out for the input signal by fitting a least squares error line to the data. The merits of this approach are it is generalisable, potentially more accurate and the computation of D_F is based on an explicit formula.

2.4 Hybrid Methods

Hybrid methods calculate the fractal dimension of 2-D surfaces using 1-D methods. This approach is based on the relationship that exists between the fractal dimensions of a surface's contours (1-D fractal curves) and the fractal dimension of the surface itself,

$D_2 = 1+D_1$ **……………..(2.4.1)**

where D_1 is the average of the fractal dimensions of each contour line and D_2 is the fractal dimension of the surface (Turner et al., 1998).

3. Merits of fractal dimension

There are many advantages of using fractal dimension over other image features. The fractal dimension is insensitive to different image transformations such as changes in local intensity, zooming, and local scaling of grey level values. Though fractal dimension is insensitive to image zooming, in practice this is valid only to a certain extent. Fractal dimension is also insensitive to multiplicative noise. If the grayscale values in a region around a pixel with coordinates *(x, y)* are multiplied by a constant factor M, then the new maximum and minimum Values are simply multiplied by the same constant *M*. If the average variation around *(x,y)* is $E_\varepsilon^{(old)}$, then the new average variation is

$\ldots E_\varepsilon^{(new)} = ME_\varepsilon^{(old)}$ **……….….(3.1)**

The FD is equal to the slope of the line that best fits the points *{log(R/$_\varepsilon$), log {(R/$_\varepsilon$)^3 ME$_\varepsilon$}) or (log(R/$_\varepsilon$), log {(R/$_\varepsilon$)^{3}E$_\varepsilon$}+log M).* The presence of the constant term *log M* does not change the slope of the line, thus the fractal dimension remains unchanged. Practically, *M* can slowly vary because it is sufficient to be fairly constant in a relatively small window (like 16*16). This implies that different regions of the image are multiplied by different constant factors, the segmentation results will not change significantly. Also, fractal dimension shows strong correlation with human judgement of surface roughness (Chaudhuri and Sarkar, 1995 ; Kasparis et al., 2001).

In Fourier Transform, as the complexity increases, the frequency domain will contain frequency components of high value. Though Fourier Transform can reconstruct or regenerate the curve or the surface with Fourier coefficients, one cannot get the impression of how complex it is. Fractal dimensions provide the measure of the complexity of curves or surfaces. Using this feature, one can get the impression of how complex or how coarse the curve or surface may be, though one cannot get the whole 'vision' of it. The fractal dimension will be higher for more complex curves and surfaces. So the two techniques could be combined to make the results more precise and more reasonable. Fractal has shown the potential of compressing data by recording the fractal dimension and some data, which could be much less than the original one and when needed it can fee restored (Lian and Cui, 1999).

4. Concept of multifractals

The fractal dimensions computed by different methods can give different answers (Turner et al., 1998). A structure can be considered as a mixture of different fractals, each one with a different value of box-counting dimension. In such a case, the conglomerate will have a dimension which is the dimension of the largest component. That means the resulting number cannot be characteristic of the mixture. In such situations, it is better to have something more like a spectrum of numbers which gives information about the distribution of fractal dimensions in a structure. This is the underlying concept in multifractals (Petgen et al., 1991; Vehel and Mignot, 1994 ; O' Neil, 1997).

5. Fractal Properties

In certain cases, especially in texture analysis the fractal dimension property alone is not sufficient to quantify a given texture. Different fractals can have the same values of fractal dimension. Hence similar fractal measures can be defined to supplement fractal dimension to uniquely define a texture.

5.1 Fractal Signature

For a pure fractal gray level image , the area at a scale e is given by

$A(\varepsilon) = F\varepsilon^{2-D}$ **…………...(5.1.1)**

where ε is the resolution of the gray levels in the image, *D* is fractal dimension and *F* is a constant. The change in measured area with changing scale is known as the fractal signature of the image (Peleg et al., 1984).

6. The Correlation Dimension and signature

Each pixel in a grey scale image can be regarded as a point in a three-dimensional space $X_t = [\ i,\ j,\ g(i, j)\ ]$ where *i* and *j* are the pixel coordinates and *g* is the grey level value at that point. For each pixel, a cube of size *23 + 1* is constructed centred at that pixel. The number of points X_i that fall inside this cube is counted for various values of δ The probability, *C(δ),* that at least one point lies within the cube can then be obtained by dividing the number of points by the cube volume.

$$C(\delta) = \_\ 1/N (2\delta + l)^3 \sum_{k=1} \ \sum_{l=1\neq k} \theta(\delta - | X_k - X_l) \ \text{.........} \ (6.1).$$

where *N* is the number of pixels in the image and θ is the Heaviside step function,

$$\theta(\xi) = \begin{cases} 1 & \xi \geq 0 \\ 0 & \xi < 0 \end{cases} \ \text{.........} \ (6.2)$$

C(δ) obeys the Richardson law with a dimension *Dc,* the correlation dimension and is given by

$$C(\delta) - c(2\delta+l)^3 -^{Dc} \ \text{......} \ (6.3)$$

The correlation signature is obtained as a plot of D_c against *3* when a single value of D_c can be computed by the normal logarithmic least squares fit or c = *C(0)* is initially computed and *C(δ)* for *δ = 1,2,* ...are subsequently computed..

7. Information dimension

If p_i is the probability of element i in the population, then information function is defined as

$$I= -\sum_{Pi}(\delta) \log P_i (\delta) \quad (7.1)$$

and the information dimension or Renyi dimension is given as (Turner et al.,1998)

$$D_1 = - \lim I / \log(\delta) = \lim (\sum p_i(\delta) \log pi (\delta))/\log(\delta) \ \text{......} \ (7.2)$$

8. Lyapunov Dimension

The Lyapunov dimension is equivalent to the correlation dimension. The Lyaponov characteristic exponents, σi, *i=1, ...,n* define the rate of exponential divergence of the perturbations and it measures the amount of instability within any dynamic system. There are as many exponents as there are dimensions in the state space of the system. So if the movement is defined by an n-dimensional map, M, there are n Lyaponov characteristic exponents, σ1, σ2 ……,σ_n. The Lyapunov dimension, D_L of a chaotic system is defined as (Turner et al., 1998)

$$D_L = l-\sigma_1/ \sigma_2 \ \text{............} \ \mathbf{(8.1)}$$

where the Lyaponov exponent are positive and $\sigma_2 > \sigma_1$

9. Lacunarity

The term lacunarity is derived from the Latin lacuna which means gap. Mandelbrot (1977) has defined suitable mathematical measures for the lacunarity of deterministic fractal sets such as the Cantor sets. These definitions are unsuitable for random fractals. Hence new definitions based on the idea of mass distribution were introduced. If a curve f(x) has negative values, then it is translated so that it is becomes non-negative. Assuming the values of / to represent mass, f is treated as a distribution of mass^ver the support of/ The lacunarity, *L,* is given by

$$L = \langle ((f / \langle f \rangle) - l)^2 \rangle = (\langle f^2 \rangle / \langle f \rangle^2 - 1 \quad \ldots\ldots\ldots (9.1)$$

$$L = \langle (f / \langle f \rangle) -l \ \rangle \ldots\ldots\ldots (9.2)$$

The equation (3.6) can be expressed as the variance divided by the square of the mean.

10. Conclusion

Among the various dimensions, fractal dimension, Information dimension and Correlation dimension are widely used in many applications. The Information and Correlation dimensions are particularly useful for data mining, since the numerator of the formula for computing Information dimension is Shannon's entropy, and correlation dimension measures the probability that two points chosen at random will be within a certain distance of each other. Changes in the information dimension correspond to the changes in the entropy and therefore point to changes in trends. Similarly, changes in the correlation dimension correspond to the changes in the distribution of points in the dataset.

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